

# LOCAL RINGS OF EMBEDDING CODEPTH AT MOST 3 HAVE ONLY TRIVIAL SEMIDUALIZING COMPLEXES

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**ABSTRACT.** We prove that a local ring  $R$  of embedding codepth at most 3 has at most two semidualizing complexes up to shift-isomorphism, namely,  $R$  itself and a dualizing  $R$ -complex if one exists.

## 1. INTRODUCTION

**Convention.** In this paper,  $R$  is a commutative noetherian ring. In this section, assume that  $(R, \mathfrak{m}, k)$  is local.

A “semidualizing”  $R$ -complex is a homologically finite  $R$ -complex  $X$  such that the natural morphism  $R \rightarrow \mathbf{R}\mathrm{Hom}_R(X, X)$  is an isomorphism in the derived category  $\mathcal{D}(R)$ . In particular, if  $X$  is a finitely generated  $R$ -module, then it is semidualizing if  $\mathrm{Hom}_R(X, X) \cong R$  and  $\mathrm{Ext}_R^{\geq 1}(X, X) = 0$ . These notions were introduced by Foxby [12] and Christensen [9] and, as special cases, recover Grothendieck’s dualizing complexes. For some indications of their usefulness, see, e.g., [5, 7, 23, 24].

In [19] we show that  $R$  has only finitely many semidualizing complexes up to shift-isomorphism, answering a question of Vasconcelos [26]. The next natural question is: how many semidualizing complexes does a given ring have up to shift-isomorphism? Progress on this question is limited, see, e.g., [8, 22, 25]. As further progress, the main result of this paper is the following, which we prove in 4.3. In the statement, the *embedding codepth* of  $R$  is  $\mathrm{ecodepth}(R) = \mathrm{edim}(R) - \mathrm{depth}(R)$ .

**Theorem A.** *Let  $R$  be a local ring that is Golod or such that  $\mathrm{ecodepth}(R) \leq 3$ . Then  $R$  has at most two distinct semidualizing complexes up to shift isomorphism, namely,  $R$  itself and a dualizing  $R$ -complex if one exists.*

The proof uses differential graded algebra techniques, as pioneered by Avramov and his collaborators. It is worth noting that one can prove the Golod case of this result using a result of Jorgensen [17, Theorem 3.1]. However, our approach is different and addresses both cases simultaneously.

**Summary.** Section 2 consists of foundational material about semidualizing objects in the DG setting. Section 3 contains versions of results from [21] for trivial extensions of DG algebras, including Theorem 3.4 which is key for our proof of Theorem A and may be of independent interest. Section 4 is mainly concerned with the proof of Theorem A.

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**Background.** We assume that the reader is familiar with many notions from the world of differential graded (DG) algebra. References on the subject include [1, 2, 4, 6, 7, 11, 13, 19, 20]. We most closely follow the conventions from [19]. For the reader's convenience, we specify some terminology and notation.

Complexes of  $R$ -modules are indexed homologically. This includes DG algebras and DG modules. Also, our DG algebras are all non-negatively graded. Given a DG  $R$ -algebra  $A$ , the underlying graded algebra for  $A$  is denoted  $A^\natural$ , and  $A$  is *homologically degree-wise noetherian* if  $H_0(A)$  is noetherian and the  $H_0(A)$ -module  $H_i(A)$  is finitely generated for all  $i \geq 0$ . When  $(R, \mathfrak{m})$  is local, we say that  $A$  is *local* if it is homologically degree-wise noetherian and the ring  $H_0(A)$  is a local  $R$ -algebra; in this case, the “augmentation ideal” of  $A$  is denoted  $\mathfrak{m}_A$ .

The derived category of DG  $A$ -modules is denoted  $\mathcal{D}(A)$ , and  $-\otimes_A^{\mathbf{L}}-$  and  $\mathbf{R}\mathrm{Hom}_A(-, -)$  are the derived functors of  $-\otimes_A-$  and  $\mathrm{Hom}_A(-, -)$ . Given an integer  $i$  and DG  $A$ -modules  $X$  and  $Y$ , we set  $\mathrm{Tor}_i^A(X, Y) := H_i(X \otimes_A^{\mathbf{L}} Y)$ , and we let  $\Sigma^i X$  denote the  $i$ th *shift* (or *suspension*) of  $X$ . Isomorphisms in  $\mathcal{D}(A)$  are identified by the symbol  $\simeq$ , and the DG modules  $X$  and  $Y$  are “shift-isomorphic”, denoted  $X \sim Y$ , if  $X \simeq \Sigma^i Y$  for some  $i \in \mathbb{Z}$ . We write  $\mathrm{id}_A(X) < \infty$  when  $X$  has a bounded semi-injective resolution; and  $\mathrm{pd}_A(X) < \infty$  means that  $X$  has a bounded semi-free resolution. When  $A$  is local and  $X$  is homologically finite, the *Poincaré series* of  $X$  is  $P_X^A(t) := \sum_{i \in \mathbb{Z}} \mathrm{len}_k(\mathrm{Tor}_i^A(k, X))t^i$  where  $k = A/\mathfrak{m}_A$ .

## 2. SEMIDUALIZING DG MODULES

**Convention.** In this section,  $A$  is a homologically degree-wise noetherian DG  $R$ -algebra.

This section contains some useful DG variations of standard results for semidualizing complexes over rings. We begin with the following definitions from [10, 13].

**Definition 2.1.** A *semidualizing* DG  $A$ -module is a homologically finite DG  $A$ -module  $X$  such that the natural homothety morphism  $R \xrightarrow{\chi_A^X} \mathbf{R}\mathrm{Hom}_A(X, X)$  is an isomorphism in  $\mathcal{D}(A)$ . When  $A$  is a ring concentrated in degree 0, these are “semidualizing  $A$ -complexes”. The set of shift-isomorphism classes (in  $\mathcal{D}(A)$ ) of semidualizing DG  $A$ -modules is denoted  $\mathfrak{S}(A)$ . A *dualizing* DG  $A$ -module is a semidualizing DG  $A$ -module  $D$  such that for every homologically finite DG  $A$ -module  $M$  the complex  $\mathbf{R}\mathrm{Hom}_A(M, D)$  is homologically finite, and the natural morphisms  $M \rightarrow \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(M, D), D)$  and  $D \otimes_A^{\mathbf{L}} M \rightarrow \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(M, D), D)$  are isomorphisms in  $\mathcal{D}(A)$ . The DG algebra  $A$  is *Gorenstein* if  $A$  is dualizing.

Next, we summarize some facts about dualizing DG  $A$ -modules.

**Fact 2.2.** Assume that  $R$  has a dualizing complex  $D^R$  and  $A^\natural$  is finitely generated as an  $R$ -module. Then  $\mathbf{R}\mathrm{Hom}_R(A, D^R)$  is a dualizing DG  $A$ -module by [13, Proposition 2.7]. Since  $\mathrm{id}_R(D^R) < \infty$ , we also have  $\mathrm{id}_A(\mathbf{R}\mathrm{Hom}_R(A, D^R)) < \infty$ . In particular, every finite dimensional DG algebra over a field has a dualizing DG module of finite injective dimension.

If  $A$  is concentrated in degree 0, the next result is well known, and our proof is not surprising; see, e.g., [9, (2.12) Corollary], [14, 2.9.1-2], and [15, Corollary 3.3].

**Lemma 2.3.** Assume that  $A$  has a dualizing DG module  $D^A$  of finite injective dimension, and let  $X$  be a semidualizing DG  $A$ -module. Then  $\mathbf{R}\mathrm{Hom}_A(X, D^A)$  is

a semidualizing DG  $A$ -module such that  $X \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(X, D^A) \simeq D^A$  in  $\mathcal{D}(A)$ . If  $R$  and  $A$  are local and either  $\mathrm{pd}_A(X) < \infty$  or  $\mathrm{id}_A(A) < \infty$ , then  $X \sim A$ .

*Proof.* By [1, Theorem 1], we have

$$X \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(X, D^A) \simeq \mathbf{RHom}_A(\mathbf{RHom}_A(X, X), D^A) \simeq \mathbf{RHom}_A(A, D^A) \simeq D^A.$$

Next, we have the following commutative diagram of DG  $A$ -module morphisms.

$$\begin{array}{ccc} A & \xrightarrow{\chi_A^{\mathbf{RHom}_A(X, D^A)}} & \mathbf{RHom}_A(\mathbf{RHom}_A(X, D^A), \mathbf{RHom}_A(X, D^A)) \\ \chi_A^{D^A} \downarrow \simeq & & \downarrow \simeq \\ \mathbf{RHom}_A(D^A, D^A) & \xrightarrow[\simeq]{\mathbf{RHom}_A(\zeta, D^A)} & \mathbf{RHom}_A(X \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(X, D^A), D^A) \end{array}$$

The morphism  $\zeta$  is the isomorphism from the beginning of this proof. It follows that  $\chi_A^{\mathbf{RHom}_A(X, D^A)}$  is an isomorphism, so  $\mathbf{RHom}_A(X, D^A)$  is a semidualizing.

Assume that  $R$  and  $A$  are local and either  $\mathrm{pd}_A(X) < \infty$  or  $\mathrm{id}_A(A) < \infty$ . Then by [1, Theorem 1] we have the following isomorphisms in  $\mathcal{D}(A)$ .

$$X \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(X, A) \simeq \mathbf{RHom}_A(\mathbf{RHom}_A(X, X), A) \simeq A.$$

It follows that  $P_X^A(t) P_{\mathbf{RHom}_A(X, A)}^A(t) = 1$ , so we have  $P_X^A(t) = t^d$  for some  $d \in \mathbb{Z}$ .

From this, we conclude that the minimal semi-free resolution  $F \xrightarrow{\simeq} X$  has  $F^\natural \simeq \Sigma^d A^\natural$ . From the Leibniz rule on  $F$ , one concludes that  $X \simeq F \simeq \Sigma^d A$ .  $\square$

The next result compares to [10, A.3. Lemma (a)].

**Lemma 2.4.** *Let  $\underline{t} = t_1, \dots, t_n$  be a sequence in the Jacobson radical of  $R$ , and set  $K = K^R(\underline{t})$ . Let  $C$  be a DG  $R$ -algebra such that  $C^\natural$  is finitely generated over  $R$ , and set  $B = K \otimes_R C$ . Then for each homologically finite DG  $C$ -module  $X$  we have  $X \in \mathfrak{S}(C)$  if and only if  $B \otimes_C^{\mathbf{L}} X \in \mathfrak{S}(B)$ .*

*Proof.* Consider the following commutative diagram of chain maps.

$$\begin{array}{ccc} B & \xrightarrow{=} & K \otimes_R C \\ \chi_B^{B \otimes_C^{\mathbf{L}} X} \downarrow & & \downarrow K \otimes \chi_C^X \\ \mathbf{RHom}_B(B \otimes_C^{\mathbf{L}} X, B \otimes_C^{\mathbf{L}} X) & & K \otimes_R \mathbf{RHom}_C(X, X) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathbf{RHom}_C(X, B \otimes_C^{\mathbf{L}} X) & & \\ = \downarrow & & \\ \mathbf{RHom}_C(X, (K \otimes_R C) \otimes_C^{\mathbf{L}} X) & \xrightarrow{\simeq} & \mathbf{RHom}_C(X, K \otimes_R^{\mathbf{L}} X) \end{array}$$

It follows that  $\chi_B^{B \otimes_C^{\mathbf{L}} X}$  is an isomorphism if and only if  $K \otimes \chi_C^X$  is an isomorphism. Since  $C$  and  $\mathbf{RHom}_C(X, X)$  are homologically degree-wise finite over  $R$ , we conclude (say, from a routine mapping cone argument) that  $K \otimes \chi_C^X$  is an isomorphism if and only if  $\chi_C^X$  is an isomorphism.  $\square$

## 3. VANISHING OF TOR OVER TRIVIAL EXTENSIONS OF DG ALGEBRAS

**Convention.** In this section,  $(R, \mathfrak{m}, k)$  is a local ring.

The point of this section is to prove that Tor-vanishing over certain trivial extensions of DG algebras implies finite projective dimension; see Theorem 3.4. We begin with a useful construction.

**Proposition 3.1.** *Let  $(A, A_+)$  be a local DG  $R$ -algebra. Let  $X$  be a homologically finite DG  $A$ -module. Then there exists a short exact sequence*

$$0 \rightarrow X' \xrightarrow{\alpha} L \xrightarrow{\pi} \tilde{X} \rightarrow 0$$

*of morphisms of DG  $A$ -modules such that  $L$  is semi-free with a finite semibasis and we have  $\tilde{X} \simeq X$  and  $\text{Im}(\alpha) \subseteq A_+L$ .*

*Proof.* Let  $F \xrightarrow{\simeq} X$  be a minimal semi-free resolution of  $X$ , and let  $E$  be a semi-basis for  $F$ . Let  $F^{(p)}$  be the semi-free DG  $A$ -submodule of  $F$  spanned by  $E_{\leq p} := \bigcup_{m \leq p} E_m$ . To be clear,  $F^{(p)\natural}$  is the graded submodule of  $F^\natural$  spanned by  $E_{\leq p}$ . Note that  $\partial^F(F^{(p)\natural}) \subseteq F^{(p)\natural}$ . To see this, by the Leibniz rule we have  $\partial_{i+j}^F(te) = \partial_i^A(t)e + (-1)^i t \partial_j^F(e)$  for each  $t \in A$  of degree  $i$  and each  $e \in F$  of degree  $j$ . If  $j \leq p$ , then the term  $\partial_i^A(t)e$  is in  $\text{Span}_A(E_{\leq p})$  by assumption, and the term  $(-1)^i t \partial_j^F(e)$  is in  $\text{Span}_A(E_{\leq p})$  by a degree argument.

Let  $s = \sup(X)$ , and set  $\tilde{X} = \tau_{\leq s}(F)$ , the “soft truncation” of  $F$ . Note that the natural morphism  $F \rightarrow \tilde{X}$  is a surjective quasiisomorphism of DG  $A$ -modules, so we have  $\tilde{X} \simeq F \simeq X$ . Next, set  $L = F^{(s)}$ , which is semi-free with a finite semibasis  $E_{\leq s}$ . Furthermore, the composition  $\pi$  of natural morphisms  $L = F^{(s)} \rightarrow F \rightarrow \tilde{X}$  is surjective because the morphism  $F \rightarrow \tilde{X}$  is surjective, the morphism  $L \rightarrow F$  is surjective in degrees  $\leq s$ , and we have  $\tilde{X}_i = 0$  for all  $i > s$ . Thus, it remains to show that  $X' := \text{Ker}(\pi) \subseteq A_+L$ .

By construction, for  $i \leq s$ , the map  $\pi_s$  is an isomorphism, so  $X'_i = 0 \subseteq A_+L$ , as desired. In degree  $s+1$ , we have  $X'_{s+1} = \text{Im}(\partial_{s+1}^F)$ , which is contained in  $(A_+F)_s$  by construction. Also, since  $A_+$  consists of elements of positive degree, we have

$$(A_+F)_s \subseteq A_+ \text{Span}_A(E_{<s}) \subseteq A_+F^{(s)}.$$

Thus, we have  $X'_{s+1} \subseteq A_+F^{(s)}$ . Lastly, for  $i > s$ , we have  $X'_i = F_i^{(s)} = (A_+F^{(s)})_i$  by a degree argument, since  $F^{(s)}$  is generated over  $A$  in degrees  $\leq s$ .  $\square$

**Corollary 3.2.** *Let  $(A, A_+)$  be a bounded local DG  $R$ -algebra. Let  $X$  and  $Y$  be homologically finite DG  $A$ -modules of infinite projective dimension over  $A$ . Assume that  $\text{Tor}_{i \gg 0}^A(X, Y) = 0$ . Then there are homologically finite DG  $A$ -modules  $X', Y'$  of infinite projective dimension over  $A$  such that  $\text{Ann}_A(A_+)X' = 0 = \text{Ann}_A(A_+)Y'$  and  $\text{Tor}_{i \gg 0}^A(X', Y') = 0$ .*

*Proof.* By Proposition 3.1, there are short exact sequences

$$0 \rightarrow X' \xrightarrow{\alpha} L \rightarrow \tilde{X} \rightarrow 0$$

$$0 \rightarrow Y' \xrightarrow{\beta} M \rightarrow \tilde{Y} \rightarrow 0$$

of morphisms of DG  $A$ -modules such that  $L$  and  $M$  are semi-free with a finite semibases, and  $\tilde{X} \simeq X$ ,  $\tilde{Y} \simeq Y$ ,  $\text{Im}(\alpha) \subseteq A_+L$ , and  $\text{Im}(\beta) \subseteq A_+M$ . The condition

$\text{Im}(\alpha) \subseteq A_+ L$  implies that

$$\text{Ann}_A(A_+)X' \cong \text{Ann}_A(A_+)\text{Im}(\alpha) \subseteq \text{Ann}_A(A_+)A_+L = 0$$

and similarly  $\text{Ann}_A(A_+)Y' = 0$ . By assumption,  $\text{pd}_A(L) < \infty$ , so the condition  $\text{pd}_A(\tilde{X}) = \text{pd}_A(X) = \infty$  implies that  $\text{pd}_A(X') = \infty$ , because of the exact sequence  $0 \rightarrow X' \rightarrow L \rightarrow \tilde{X} \rightarrow 0$ . And similarly  $\text{pd}_A(Y') = \infty$ . Finally, since  $A$  is bounded and  $\text{pd}_A(L)$  is finite, the fact that  $Y$  is homologically bounded implies that  $\text{Tor}_{i \gg 0}^A(L, Y) = 0$ . By assumption, we have  $\text{Tor}_{i \gg 0}^A(\tilde{X}, Y) \cong \text{Tor}_{i \gg 0}^A(X, Y) = 0$  so the above exact sequence implies that  $\text{Tor}_{i \gg 0}^A(X', Y) = 0$ . Similarly, we deduce  $\text{Tor}_{i \gg 0}^A(X', Y') = 0$ , as desired.  $\square$

Compare the next two results to [21, Lemma 3.2 and Theorem 3.1].

**Lemma 3.3.** *Let  $(B, \mathfrak{m}_B, k)$  be a local DG  $R$ -algebra. Set  $A = B \ltimes \Sigma^n k$  for some  $n \geq 0$ , and let  $\mathfrak{m}_A$  be the augmentation ideal of  $A$ . Let  $x$  be a generator for the DG ideal  $0 \oplus \Sigma^n k \subseteq A$ . Let  $X, Y$  be DG  $B$ -modules, i.e., DG  $A$ -modules such that  $xX = 0 = xY$ . Then for all  $i \in \mathbb{Z}$  we have  $R$ -module isomorphisms*

$$\text{Tor}_i^A(X, Y) \cong \text{Tor}_i^B(X, Y) \bigoplus \left( \bigoplus_{p+q=i-n-1} \text{Tor}_p^A(X, k) \otimes_k \text{Tor}_q^B(k, Y) \right).$$

*Proof.* Let  $s: L \xrightarrow{\sim} B$  be a semi-free resolution over  $A$ . Let  $p: A \rightarrow B$  be the natural surjection, and let  $\tilde{p}: A \rightarrow L$  be a lift of  $p$ . Hence the following diagram of morphisms of DG  $A$ -modules

$$\begin{array}{ccc} A & \xrightarrow{\tilde{p}} & L \\ & \searrow p & \downarrow \simeq s \\ & & B \end{array}$$

commutes up to homotopy. Apply  $X \otimes_A -$  to obtain the next diagram of morphisms of DG  $B$ -modules

$$\begin{array}{ccc} X \otimes_A A & \xrightarrow{X \otimes \tilde{p}} & X \otimes_A L \\ & \searrow \cong & \downarrow X \otimes s \\ & & X \otimes_A B \end{array}$$

that commutes up to homotopy. Note that  $X \otimes p$  is an isomorphism since  $xX = 0$ . Also, the chain map  $X \otimes_A \tilde{p}$  represents the morphism  $X \otimes_A^{\mathbf{L}} p: X \otimes_A^{\mathbf{L}} A \rightarrow X \otimes_A^{\mathbf{L}} B$  in  $\mathcal{D}(B)$ . It follows that  $X \otimes_A^{\mathbf{L}} p$  has a left-inverse in  $\mathcal{D}(B)$ , so we have the first isomorphism (in  $\mathcal{D}(B)$ ) in the next sequence:

$$\begin{aligned} X \otimes_A^{\mathbf{L}} B &\simeq (X \otimes_A^{\mathbf{L}} A) \bigoplus (X \otimes_A^{\mathbf{L}} (\Sigma x A)) \\ &\simeq (X \otimes_A^{\mathbf{L}} A) \bigoplus (X \otimes_A^{\mathbf{L}} (\Sigma^{n+1} k)) \\ &\simeq (X \otimes_A^{\mathbf{L}} A) \bigoplus (\Sigma^{n+1} X \otimes_A^{\mathbf{L}} k). \end{aligned}$$

The second isomorphism is from the assumption  $\mathfrak{m}_A x = 0$ . (Note that this isomorphism is in  $\mathcal{D}(A)$ , hence also in  $\mathcal{D}(B)$  via the split injection  $B \rightarrow A$ .) Now, we

apply  $-\otimes_B^{\mathbf{L}} Y$  to these isomorphisms to conclude that

$$\begin{aligned} X \otimes_A^{\mathbf{L}} Y &\simeq (X \otimes_A^{\mathbf{L}} B) \otimes_B^{\mathbf{L}} Y \\ &\simeq ((X \otimes_A^{\mathbf{L}} A) \otimes_B^{\mathbf{L}} Y) \bigoplus (\Sigma^{n+1}(X \otimes_A^{\mathbf{L}} k) \otimes_B^{\mathbf{L}} Y) \\ &\simeq (X \otimes_B^{\mathbf{L}} Y) \bigoplus (\Sigma^{n+1}(X \otimes_A^{\mathbf{L}} k) \otimes_k^{\mathbf{L}} (k \otimes_B^{\mathbf{L}} Y)). \end{aligned}$$

Apply  $H_i(-)$  to obtain the first isomorphism in the next sequence:

$$\begin{aligned} \mathrm{Tor}_i^A(X, Y) &\cong \mathrm{Tor}_i^B(X, Y) \bigoplus H_i(\Sigma^{n+1}(X \otimes_A^{\mathbf{L}} k) \otimes_k^{\mathbf{L}} (k \otimes_B^{\mathbf{L}} Y)) \\ &\cong \mathrm{Tor}_i^B(X, Y) \bigoplus H_{i-n-1}((X \otimes_A^{\mathbf{L}} k) \otimes_k^{\mathbf{L}} (k \otimes_B^{\mathbf{L}} Y)) \\ &\cong \mathrm{Tor}_i^B(X, Y) \bigoplus \left( \bigoplus_{p+q=i-n-1} H_p(X \otimes_A^{\mathbf{L}} k) \otimes_k H_q(k \otimes_B^{\mathbf{L}} Y) \right) \\ &\cong \mathrm{Tor}_i^B(X, Y) \bigoplus \left( \bigoplus_{p+q=i-n-1} \mathrm{Tor}_p^A(X, k) \otimes_k \mathrm{Tor}_q^B(k, Y) \right). \end{aligned}$$

The third isomorphism is from the Künneth formula.  $\square$

**Theorem 3.4.** *Let  $(B, B_+, k)$  be a bounded local DG  $R$ -algebra. Set  $A = B \ltimes \Sigma^n k$  for some  $n \geq 1$ . Let  $x$  be a generator for the DG ideal  $0 \oplus \Sigma^n k \subseteq A$ . Let  $X$  and  $Y$  be non-zero homologically finite DG  $A$ -modules such that  $\mathrm{Tor}_{i \gg 0}^A(X, Y) = 0$ . Then  $\mathrm{pd}_A(X) < \infty$  or  $\mathrm{pd}_A(Y) < \infty$ .*

*Proof.* Assume by way of contradiction that  $\mathrm{pd}_A(X) = \infty = \mathrm{pd}_A(Y)$ . Thus, Corollary 3.2 provides homologically finite DG  $A$ -modules  $X', Y'$  of infinite projective dimension such that  $\mathrm{Ann}_A(A_+)X' = 0 = \mathrm{Ann}_A(A_+)Y'$  and such that  $\mathrm{Tor}_{i \gg 0}^A(X', Y') = 0$ . Thus, we may replace  $X$  and  $Y$  by  $X'$  and  $Y'$  to assume that  $\mathrm{Ann}_A(A_+)X = 0 = \mathrm{Ann}_A(A_+)Y$ . In particular, we have  $xX = 0 = xY$ .

Since  $\mathrm{Tor}_{i \gg 0}^A(X, Y) = 0$ , Lemma 3.3 implies that

$$\bigoplus_{p+q=i-n-1} \mathrm{Tor}_p^A(X, k) \otimes_k \mathrm{Tor}_q^B(k, Y) = 0 \quad (3.4.1)$$

for all  $i \gg 0$ . The fact that  $\mathrm{pd}_A(Y)$  is infinite implies that  $Y \neq 0$ . Since  $Y$  is homologically finite, Nakayama's Lemma implies that there is an integer  $q_0$  such that  $\mathrm{Tor}_{q_0}^B(k, Y) \neq 0$ . Thus, equation (3.4.1) for  $i \gg 0$  implies that  $\mathrm{Tor}_{p \gg 0}^A(X, k) = 0$ , contradicting the assumption  $\mathrm{pd}_A(X) = \infty$ .  $\square$

#### 4. THE NUMBER OF SEMIDUALIZING COMPLEXES FOR EMBEDDING CODEPTH 3

**Convention.** In this section,  $(R, \mathfrak{m}, k)$  is a local ring.

The purpose of this section is to prove Theorem A from the introduction, which we do in 4.3. Our main tool for this is the following result.

**Theorem 4.1.** *Assume that  $R$  admits a dualizing complex, and let  $(B, B_+, k)$  be a bounded homologically finite local DG  $R$ -algebra. Set  $A = B \ltimes W$  for some non-zero finitely generated positively graded  $k$ -vector space  $W$ . Then  $|\mathfrak{S}(A)| \leq 2$ .*

*Proof.* Fact 2.2 implies that  $A$  has a dualizing DG module  $D^A$  of finite injective dimension. Set  $(-)^{\dagger} = \mathbf{R}\mathrm{Hom}_A(-, D^A)$ . It suffices to show that for every semidualizing DG  $A$ -module  $X$  we have  $X \sim A$  or  $X \sim D^A$ .

Case 1:  $W = \Sigma^n k$  for some  $n \geq 1$ . The isomorphism  $X \otimes_A^{\mathbf{L}} X^{\dagger} \simeq D^A$  from Lemma 2.3 implies that  $\mathrm{Tor}_{i \gg 0}^A(X, X^{\dagger}) = 0$ . By Theorem 3.4 either  $\mathrm{pd}_A(X) < \infty$  or  $\mathrm{pd}_A(X^{\dagger}) < \infty$ . And by Lemma 2.3 we have  $X \sim A$  or  $X^{\dagger} \sim A$ . If  $X^{\dagger} \sim A$ , then by definition of  $D^A$  we have  $X \simeq X^{\dagger\dagger} \sim A^{\dagger} \simeq D^A$ , as desired.

Case 2: General case. Since  $W \neq 0$ , write  $W = W' \oplus \Sigma^n k$  for some  $n \geq 1$ , and set  $B' := B \ltimes W'$ . It follows that  $A \cong B' \ltimes \Sigma^n k$ , so the assertion follows from the previous case.  $\square$

**Remark 4.2.** Let  $\underline{t} = t_1, \dots, t_n$  be a minimal generating set for  $\mathfrak{m}$ , and consider the Koszul complex  $K = K^R(\underline{t})$ . From [3, (1.2)] (or [4, (2.8)]) there is a finite-dimensional DG  $k$ -algebra  $A$  that is linked to  $K$  by a sequence of quasiisomorphisms of DG algebras. As in [19, 5.4 (Proof of Theorem A)], one has an injection  $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}(\widehat{R})$  and bijections  $\mathfrak{S}(\widehat{R}) \xrightarrow{\cong} \mathfrak{S}(K) \xrightarrow{\cong} \mathfrak{S}(A)$ .

By definition (or by a result of Golod [16]),  $R$  is Golod if and only if  $A$  is of the form  $k \ltimes W$  for some finitely generated positively graded  $k$ -vector space  $W$ .

Assume for the rest of this remark that  $c = \mathrm{ecodepth}(R) \leq 3$ . Up to isomorphism, the algebra  $A$  has  $\partial^A = 0$  and is in one of the classes described in the next table, copied from [3, 1.3].

Class	$A$	$B$	$C$	$D$
<b>C</b> ( $c$ )	$B$	$\bigwedge_k \Sigma k^c$		
<b>S</b>	$B \ltimes W$	$k$		
<b>T</b>	$B \ltimes W$	$C \ltimes \Sigma(C/C_{\geq 2})$	$\bigwedge_k \Sigma k^2$	
<b>B</b>	$B \ltimes W$	$C \ltimes \Sigma C_+$	$\bigwedge_k \Sigma k^2$	
<b>G</b> ( $r$ )	$B \ltimes W$	$C \ltimes \mathrm{Hom}_k(C, \Sigma^3 k)$	$k \ltimes \Sigma k^r$	
<b>H</b> ( $p, q$ )	$B \ltimes W$	$C \otimes_k D$	$k \ltimes (\Sigma k^p \oplus \Sigma^2 k^q)$	$k \ltimes \Sigma k$

Here,  $W$  is a finitely generated positively graded  $k$ -vector space such that  $B_+ W = 0$ . Note that if  $R$  is not regular, then  $\sum_i (-1)^i \mathrm{len}_k(A_i) = \sum_i (-1)^i \mathrm{len}_k(H_i(K)) = 0$ .

**4.3 (Proof of Theorem A).** Continue with the notation from Remark 4.2. Recall that  $\widehat{R}$  has a dualizing complex  $D^{\widehat{R}}$ , and that  $R$  is Gorenstein if and only if  $D^{\widehat{R}} \sim \widehat{R}$ . Hence, it suffices to show that  $|\mathfrak{S}(A)| \leq 2$ .

Assume for this paragraph that  $R$  is Golod. Then we have  $A \cong k \ltimes W$  for some finitely generated positively graded  $k$ -vector space  $W$ . If  $W = 0$ , then  $A \cong k$  which is a commutative local Gorenstein ring, hence  $|\mathfrak{S}(A)| = 1$  in this case. If  $W \neq 0$ , then  $|\mathfrak{S}(A)| \leq 2$  by Theorem 4.1.

Assume for the rest of the proof that  $c = \mathrm{ecodepth}(R) \leq 3$ . We analyze the classes from Remark 4.2. Note that if  $R$  is Gorenstein, then  $|\mathfrak{S}(R)| = 1$ . Also, if  $W \neq 0$ , then the conclusion  $|\mathfrak{S}(A)| \leq 2$  follows from Theorem 4.1. Thus, we assume for the rest of the proof that  $W = 0$ .

If  $R$  is in the class **C**( $c$ ), then  $R$  is a complete intersection (hence Gorenstein), which has already been treated. If  $R$  is in the class **S**, then  $R$  is Golod, which has also already been treated.

(Class **T**) In this case the algebra  $C$  has the form  $0 \rightarrow k \rightarrow k^2 \rightarrow k \rightarrow 0$ . In particular,  $C_{\geq 2} = \Sigma^2 k$ . It follows that

$$\sum_i (-1)^i \operatorname{len}_k(B_i) = \sum_i (-1)^i \operatorname{len}_k(C_i) + \sum_{i < 2} (-1)^{i-1} \operatorname{len}_k(C_i) = 1 \neq 0.$$

Since we know that

$$\begin{aligned} 0 &= \sum_i (-1)^i \operatorname{len}_k(A_i) \\ &= \sum_i (-1)^i \operatorname{len}_k(B_i) + \sum_i (-1)^i \operatorname{len}_k(W_i) \\ &= 1 + \sum_i (-1)^i \operatorname{len}_k(W_i) \end{aligned}$$

we have  $W \neq 0$ , which is a case we have already treated.

(Class **B**) As in the previous case, one has  $W \neq 0$ , which has already been treated.

(Class **G(r)**) The DG  $C$ -module  $\operatorname{Hom}_k(C, \Sigma^3 k)$  is dualizing, so the algebra

$$A \cong B = C \ltimes \operatorname{Hom}_k(C, \Sigma^3 k)$$

is Gorenstein by [18, Theorem 2.2]. It follows from [13, Theorem III] that  $A \simeq \operatorname{Hom}_k(A, \Sigma^3 k)$ . Since  $\operatorname{Hom}_k(A, \Sigma^3 k)$  is bounded and semi-injective over  $A$ , we conclude that  $\operatorname{id}_A(A) < \infty$ . Thus, we have  $|\mathfrak{S}(A)| = 1$  by Lemma 2.3.

(Class **H(p,q)**) We first note that  $|\mathfrak{S}(C)| \leq 2$ . Indeed, if  $p = q = 0$ , then we have  $C = k$  so  $|\mathfrak{S}(C)| = 1$ , as above; if  $p \neq 0$  or  $q \neq 0$ , then this follows from Theorem 4.1. Hence, it remains to show that  $|\mathfrak{S}(B)| = |\mathfrak{S}(C)|$ . For this, consider the map  $\mathfrak{S}(C) \rightarrow \mathfrak{S}(B)$  defined by  $X \mapsto B \otimes_C^L X$ . This is well-defined by Lemma 2.4, and it is bijective by [20, Theorems 3.4 and 3.11]. (This uses the fact that  $D$  is isomorphic to the trivial Koszul complex  $K^k(0)$ .)  $\square$

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